Achievable efficiencies for probabilistically cloning the states

2004 J. Phys. A: Math. Gen. 37 3211

(http://iopscience.iop.org/0305-4470/37/9/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 3211–3223 PII: S0305-4470(04)71686-8

Achievable efficiencies for probabilistically cloning the states

T Gao[1,2], F L Yan[3,4] and Z X Wang[1]

[1] Department of Mathematics, Capital Normal University, Beijing 100037, People's Republic of China
[2] College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050016, People's Republic of China
[3] Department of Physics, Hebei Normal University, Shijiazhuang 050016, People's Republic of China
[4] CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China

E-mail: gaoting@heinfo.net

Received 10 November 2003
Published 18 February 2004
Online at stacks.iop.org/JPhysA/37/3211 (DOI: 10.1088/0305-4470/37/9/011)

Abstract

We present an example of quantum computational tasks whose performance is enhanced if we distribute quantum information using quantum cloning. Furthermore, we give achievable efficiencies for probabilistically cloning the quantum states used in implemented tasks for which cloning provides some enhancement in performance.

PACS number: 03.67.Lx

1. Introduction

Cloning is a type of quantum information processing tool. In 1982 Wootters and Zurek [1] and Dieks [2] independently discovered the no-cloning theorem, one of the first results stressing the peculiarities of quantum information. They showed that unlike classical information, it is impossible to make perfect copies of an unknown quantum state, i.e. qubits cannot be copied. Since then quantum cloning has been studied intensively, and much effort has been put into developing optimal cloning processes [3–14]. There are two main approaches to quantum cloning. The first one consists in using ancillary quantum systems and a global unitary operation to obtain multiple imperfect clones of a given, unknown quantum state. These universal quantum cloning machines (UQCMs) were first invented by Bužek and Hillery [3] and developed by other authors [4–12]. The second kind of cloning procedure first designed by Duan and Guo [13, 14] is nondeterministic, consisting in adding an ancilla, performing unitary operations and measurements, with a postselection of the measurement results. The resulting clones are perfect, but the procedure only succeeds with a certain probability $p < 1$, which depends on the particular set of states that we are trying to clone. Recently, Galvão and Hardy discuss how quantum information distribution implemented with different types of quantum cloning procedures can improve the performance of some quantum computation tasks [15]. Unfortunately, in the second example they obtained the achievable efficiencies for probabilistically cloning states by a numerical search. Evidently, the numerical result is not an exact solution and this is what originally motivated the present work.

Our purpose in this paper is twofold. First we present an example of quantum computation tasks whose performance is enhanced if we distribute quantum information using quantum cloning. The second purpose of the paper is to provide achievable efficiencies for probabilistically cloning the states [15] used in implemented tasks for which cloning provides some enhancement in performance.

2. An example with probabilistic cloning

In this section, we give an example of quantum computation tasks that can be better performed if we make use of quantum cloning. The task relies on state-dependent probabilistic quantum cloning discussed by Duan and Guo [13, 14]. Now we present our example by generalizing the second example of [15] in which they discussed the functions that take two bits to one bit, to the case of three bits to one bit.

The quantum computational task is as follows. Suppose that we are given three quantum black-boxes. What each black-box does is to accept four two-level quantum systems as an input and apply a unitary operator to it, producing the evolved state as the output. We take the black-boxes to consist of arbitrary quantum circuits that query a given function only once. The query of function f_i is the unitary that performs $|x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f_i(x)\rangle$, where the symbol $\oplus$ represents the bitwise XOR operation. Our task will involve determining two functionals, one depending only on f_0 and f_1, and the other on f_0 and f_2. We will prove that cloning offers an advantage which cannot be matched by any approach that does not resort to quantum cloning.

In order to precisely state our task, we start by considering all functions h_i which take three bits to one bit. We may represent each such function with eight bits $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ and a_8, writing $h_{a_1a_2a_3a_4a_5a_6a_7a_8}$ to stand for the function h such that $h(000) = a_1$, $h(001) = a_2$, $h(010) = a_3$, $h(011) = a_4$, $h(100) = a_5$, $h(101) = a_6$, $h(110) = a_7$, $h(111) = a_8$. Now we define some sets of functions that will be useful in stating our task:

$$S_{f_0} = \{h_{01000000}, h_{00110011}, h_{11000011}\}$$
$$S_1 = \{h_{01000000}, h_{10110000}, h_{10001100}, h_{00100110}, h_{00010101}, h_{10000011}, h_{00101001}, h_{00011010}\}$$
$$S_2 = \{h_{00000000}, h_{00001111}, h_{01010101}, h_{00110011}, h_{10011001}, h_{11000011}, h_{01101001}, h_{10100101}\}$$
$$S_{f_{12}} = S_1 \cup S_2$$
$$S_{00000000} = \{h_{00000000}, h_{11111111}\} \qquad S_{00001111} = \{h_{00001111}, h_{11110000}\}$$
$$S_{01010101} = \{h_{01010101}, h_{10101010}\} \qquad S_{00110011} = \{h_{00110011}, h_{11001100}\}$$
$$S_{10011001} = \{h_{10011001}, h_{01100110}\} \qquad S_{11000011} = \{h_{11000011}, h_{00111100}\}$$
$$S_{01101001} = \{h_{01101001}, h_{10010110}\} \qquad S_{10100101} = \{h_{10100101}, h_{01011010}\}$$
$$S_f = S_{00000000} \cup S_{00001111} \cup S_{01010101} \cup S_{00110011} \cup S_{10011001} \cup S_{11000011} \cup S_{01101001} \cup S_{10100101}.$$

Now we first randomly choose a function $f_0 \in S_{f_0}$, then two other functions f_1 and f_2 are picked from the set $S_{f_{12}}$, also at random but satisfying

$$f_0 \oplus f_1 \qquad f_0 \oplus f_2 \in S_f. \tag{1}$$

Here the symbol $\oplus$ is addition modulo 2. The task will be to find in which of the eight sets $S_{00000000}$, $S_{00001111}$, $S_{01010101}$, $S_{00110011}$, $S_{10011001}$, $S_{11000011}$, $S_{01101001}$ and $S_{10100101}$ lie each

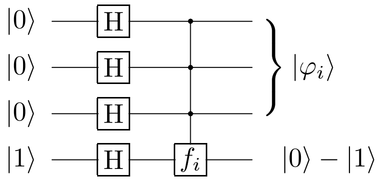

Figure 1. If function f_i is guaranteed to be either in set S_1 or in S_2, then this quantum circuit can be used to distinguish between the eight possibilities in each set. We can determine f_i by measuring the final state $|\varphi_i\rangle = \frac{1}{2\sqrt{2}}\sum_{x=000}^{111}(-1)^{f_i(x)}|x\rangle$ in one of two orthogonal bases, depending on which set contains f_i. Here H operations are Hadamard gates.

of the functions $f_0 \oplus f_1$ and $f_0 \oplus f_2$, applying quantum circuits that query f_0, f_1 and f_2 at most once each. Our score will be given by the average probability of successfully guessing both correctly.

2.1. Score without cloning

Now we will give the attainable score if we do not resort to cloning. Just as [15] the best no-cloning strategy goes as follows. Firstly, from the constraints given by equation (1) we note that both f_1 and f_2 must be in S_1 if $f_0 = h_{01000000}$, and f_1 and f_2 must belong to S_2 if f_0 is either $h_{00110011}$ or $h_{11000011}$. Since f_0 were drawn from a uniformly random distribution, the probability of both f_1 and f_2 in S_2 is $2/3$. Assume that it is the case, then we can discriminate between the two possibilities for f_0 with a single, classical function call. Furthermore, by using the quantum circuit in figure 1 twice (once each with f_1 and f_2) we can distinguish the eight possibilities for functions f_1 and f_2.

This happens because depending on which function in S_2 was queried, this quantum circuit results in one of the eight orthogonal states

$$|\varphi_i\rangle = \frac{1}{2\sqrt{2}}\sum_{x=000}^{111}(-1)^{f_i(x)}|x\rangle. \tag{2}$$

This allows us to determine functions f_0, f_1 and f_2 correctly with probability $p = 2/3$, in which case we can determine which sets contain $f_0 \oplus f_1$ and $f_0 \oplus f_2$ and accomplish our task. Even in the case where the initial assumption about f_0 was wrong, we may still have guessed the right sets by chance; the chances of getting both right this way are $1/64$. Thus, the best no-cloning average score is

$$p_1 = \tfrac{2}{3} + \tfrac{1}{3} \times \tfrac{1}{64} = 0.671\,875. \tag{3}$$

2.2. Score with cloning

Next we will prove that we can do much better than that with quantum cloning. The idea is similar to [15], that is, to devise a quantum circuit that queries function f_0 only once, makes two clones of the resulting state, and then queries functions f_1 and f_2, one in each branch of the computation. Since we have some information about the state produced by one query of f_0, the probabilistic cloning machines investigated by Duan and Guo [13] will suit this task better.

The quantum circuit that we use to solve this problem is depicted in figure 2. Immediately after querying function f_0, we have one of three possible linearly independent states (each

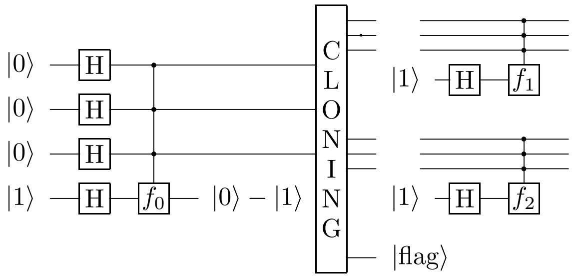

Figure 2. The cloning procedure in this circuit is probabilistic. After the cloning process we can measure a 'flag' subsystem and know whether the cloning was successful or not. If the cloning is successful, we let the clones go through the rest of the circuit, yielding output states $|\varphi_i\rangle = \frac{1}{2\sqrt{2}}\sum_{x=000}^{111}(-1)^{f_0(x)+f_i(x)}|x\rangle$ $(i = 1, 2)$. These states can be measured in the basis defined by equations (9)–(16) to unambiguously decide which of the eight sets $S_{00000000}$, $S_{00001111}$, $S_{01010101}$, $S_{00110011}$, $S_{10011001}$, $S_{11000011}$, $S_{01101001}$, $S_{10100101}$ contains $f_0 \oplus f_i$.

corresponding to one of the possible f_0):

$$|\Psi_1\rangle \equiv |h_{01000000}\rangle \equiv \frac{1}{2\sqrt{2}}[|000\rangle - |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle] \tag{4}$$

$$|\Psi_2\rangle \equiv |h_{00110011}\rangle \equiv \frac{1}{2\sqrt{2}}[|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle] \tag{5}$$

$$|\Psi_3\rangle \equiv |h_{11000011}\rangle \equiv \frac{1}{2\sqrt{2}}[-|000\rangle - |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle]. \tag{6}$$

The probabilistic cloning machines with different cloning efficiencies (defined as the probability of cloning successfully) for each of states (4)–(6) will be constructed. From theorem 2 in [13] we obtain the following exact achievable efficiencies,

$$\gamma_1 \equiv \gamma(|h_{01000000}\rangle) = \frac{7}{127} \tag{7}$$

$$\gamma_2 \equiv \gamma(|h_{00110011}\rangle) = \gamma_3 \equiv \gamma(|h_{11000011}\rangle = \frac{112}{127} \tag{8}$$

which will be shown in the next section.

After the cloning process a measurement on a 'flag' subsystem is performed and the result will tell us whether the cloning was successful or not. For this particular cloning process, the probability of success is, on average, $P_{\text{success}} = (\gamma_1 + \gamma_2 + \gamma_3)/3 = \frac{77}{127}$. If it was successful, then each of the cloning branches goes through the second part of the circuit in figure 2, to yield one of the eight orthogonal states,

$$|h_{00000000}\rangle \equiv \frac{1}{2\sqrt{2}}[|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle] \tag{9}$$

$$|h_{00001111}\rangle \equiv \frac{1}{2\sqrt{2}}[|000\rangle + |001\rangle + |010\rangle + |011\rangle - |100\rangle - |101\rangle - |110\rangle - |111\rangle] \tag{10}$$

$$|h_{01010101}\rangle \equiv \frac{1}{2\sqrt{2}}[|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle] \tag{11}$$

$$|h_{00110011}\rangle \equiv \frac{1}{2\sqrt{2}}[|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle] \tag{12}$$

$$|h_{10011001}\rangle \equiv \frac{1}{2\sqrt{2}}[-|000\rangle + |001\rangle + |010\rangle - |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle] \quad (13)$$

$$|h_{11000011}\rangle \equiv \frac{1}{2\sqrt{2}}[-|000\rangle - |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle] \quad (14)$$

$$|h_{01101001}\rangle \equiv \frac{1}{2\sqrt{2}}[|000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle] \quad (15)$$

$$|h_{10100101}\rangle \equiv \frac{1}{2\sqrt{2}}[-|000\rangle + |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle] \quad (16)$$

which can be discriminated unambiguously. Therefore, if the cloning process is successful, we manage to accomplish our task.

However, the cloning process may fail with probability $(1 - P_{\text{success}})$. If this happens, it is more likely to be $h_{01000000}$ than the other two, because of the relatively low cloning efficiency for the state in equation (4), in relation to the states in equations (5) and (6) (see equations (7) and (8)). If we then guess that $f_0 = h_{01000000}$, we will be right with probability

$$p_{01000000} = \frac{(1-\gamma_1)}{(1-\gamma_1)+(1-\gamma_2)+(1-\gamma_3)} = \frac{4}{5}. \quad (17)$$

What is more, we are still free to design quantum circuits to obtain information about f_1 and f_2, since at this stage we still have not queried them. Given our guess that $f_0 = h_{01000000}$, only the eight functions in S_1 can be candidates for f_1 and f_2, because of the constraints given by equation (1). These eight possibilities can be discriminated unambiguously by running a circuit like that of figure 1 twice, once with f_1 and once with f_2. The circuit produces one of eight orthogonal states, each corresponding to one of the eight possibilities for f_i. Therefore if our guess that $f_0 = h_{01000000}$ was correct, we are able to find the correct f_1 and f_2 and therefore accomplish our task. In the case that $f_0 \neq h_{01000000}$ after all, we may still have guessed the right sets by chance; a simple analysis shows that this will happen with probability 1/64.

The above considerations lead to an overall probability of success given by

$$\begin{aligned} p_2 &= P_{\text{success}} + (1 - P_{\text{success}})\left[p_{01000000} + (1 - p_{01000000})\frac{1}{64}\right] \\ &= \frac{22 + 21(\gamma_2+\gamma_3)}{64} \\ &= \frac{3749}{4064} \\ &\simeq 0.922\,49 \\ &> p_1 = 0.671\,875 \end{aligned} \quad (18)$$

thus showing that this cloning approach is more efficient than the previous one, which does not use cloning.

2.3. Exact achievable efficiencies

Here we present the analytic solution of achievable efficiencies for cloning the state equations (4)–(6). As stated above we use $\gamma_1 \equiv \gamma(|h_{01000000}\rangle)$, $\gamma_2 \equiv \gamma(|h_{00110011}\rangle)$, $\gamma_3 \equiv \gamma(|h_{11000011}\rangle)$ to express the achievable efficiencies, and let $|P^{(1)}\rangle, |P^{(2)}\rangle, |P^{(3)}\rangle$ be normalized states of the flag P. P_{ij} denotes the inner product $\langle P^{(i)}|P^{(j)}\rangle$ between $|P_i\rangle$ and $|P_j\rangle$, $i, j = 1, 2, 3$. Clearly, $|P_{ij}| \leqslant 1$. Suppose the 3×3 matrices $X^{(1)} = [\langle\Psi_i|\Psi_j\rangle]$, $X_P^{(2)} = [\langle\Psi_i|\Psi_j\rangle^2 P_{ij}]$

and the diagonal efficiency matrix $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3)$, then

$$X^{(1)} - \sqrt{\Gamma} X_P^{(2)} \sqrt{\Gamma^+} = \begin{pmatrix} 1 & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \frac{\sqrt{\gamma_1\gamma_2}}{16} P_{12} & \frac{\sqrt{\gamma_1\gamma_3}}{16} P_{13} \\ \frac{\sqrt{\gamma_1\gamma_2}}{16} P_{12}^* & \gamma_2 & 0 \\ \frac{\sqrt{\gamma_1\gamma_3}}{16} P_{13}^* & 0 & \gamma_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\gamma_1 & -\frac{1}{4} - \frac{\sqrt{\gamma_1\gamma_2}}{16} P_{12} & \frac{1}{4} - \frac{\sqrt{\gamma_1\gamma_3}}{16} P_{13} \\ -\frac{1}{4} - \frac{\sqrt{\gamma_1\gamma_2}}{16} P_{12}^* & 1-\gamma_2 & 0 \\ \frac{1}{4} - \frac{\sqrt{\gamma_1\gamma_3}}{16} P_{13}^* & 0 & 1-\gamma_3 \end{pmatrix}.$$

Theorem 2 of [13] provides us with inequalities

$$1 - \gamma_1 \geqslant 0 \tag{19}$$

$$(1-\gamma_1)(1-\gamma_2) - \left|\tfrac{1}{4} + \tfrac{1}{16}\sqrt{\gamma_1\gamma_2} P_{12}\right|^2 \geqslant 0 \tag{20}$$

$$(1-\gamma_1)(1-\gamma_2)(1-\gamma_3) - (1-\gamma_3)\left|\tfrac{1}{4} + \tfrac{1}{16}\sqrt{\gamma_1\gamma_2} P_{12}\right|^2 - (1-\gamma_2)\left|\tfrac{1}{4} - \tfrac{1}{16}\sqrt{\gamma_1\gamma_3} P_{13}\right|^2 \geqslant 0 \tag{21}$$

which allow us to derive achievable efficiencies for the probabilistic cloning process. According to the rule stated in the above section (see equation (18)) the overall probability (score) of success with the help of probabilistic cloning is given by

$$\begin{aligned} p_2 &= p_{\text{success}} + (1 - p_{\text{success}})\left[p_{01000000} + (1 - p_{01000000})\frac{1}{64}\right] \\ &= \frac{\gamma_1+\gamma_2+\gamma_3}{3} + \left(1 - \frac{\gamma_1+\gamma_2+\gamma_3}{3}\right)\left[\frac{1-\gamma_1}{3-\gamma_1-\gamma_2-\gamma_3} + \left(1 - \frac{1-\gamma_1}{3-\gamma_1-\gamma_2-\gamma_3}\right)\frac{1}{64}\right] \\ &= [22 + 21(\gamma_2+\gamma_3)]/64. \end{aligned} \tag{22}$$

From above equation we know that we should find the maximum of $\gamma_2 + \gamma_3$ satisfying equations (19)–(21).

In the following, we show that the maximum of $\gamma_2 + \gamma_3$ must be greater than or equal to $\frac{224}{127}$. We consider the case $\gamma_2 = \gamma_3$. In this case, there is

$$(1-\gamma_1)(1-\gamma_2) - \left|\tfrac{1}{4} + \tfrac{1}{16}\sqrt{\gamma_1\gamma_2} P_{12}\right|^2 - \left|\tfrac{1}{4} - \tfrac{1}{16}\sqrt{\gamma_1\gamma_2} P_{13}\right|^2 \geqslant 0 \tag{23}$$

which implies that

$$\tfrac{7}{8} - qx + sx^2 \geqslant y \geqslant 2x \geqslant 0 \tag{24}$$

where $P_{12} = a + b\mathrm{i}$, $P_{13} = c + d\mathrm{i}$, $q = \frac{1}{32}(a-c)$, $s = 1 - \frac{1}{256}(a^2+b^2+c^2+d^2)$, $y = \gamma_1 + \gamma_2$, and $x = \sqrt{\gamma_1\gamma_2}$. It is not difficult to prove that

$$\frac{127}{128} \leqslant s \leqslant 1 \qquad -\frac{1}{16} \leqslant q \leqslant \frac{1}{16}. \tag{25}$$

Since $\frac{7}{8} - q + s \leqslant 2$, and $0 \leqslant x \leqslant 1$, $y = \frac{7}{8} - qx + sx^2$ and $y = 2x$ have one intersection point

$$(x_0, y_0) = \left(\frac{2 + q - \sqrt{(2+q)^2 - \frac{7}{2}s}}{2s}, \frac{2 + q - \sqrt{(2+q)^2 - \frac{7}{2}s}}{s}\right).$$

The region in the x–y plane and the region in the q–s plane governed by equation (24) are the shaded areas in figure 3 and in figure 4 respectively.

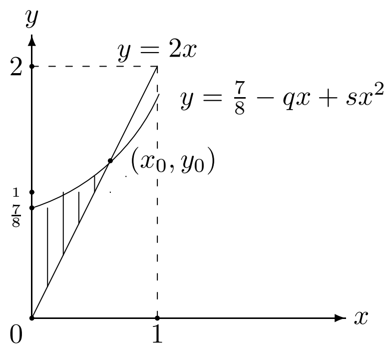

Figure 3. The (x, y) region. Note $\frac{7}{8} - q + s \leqslant 2$.

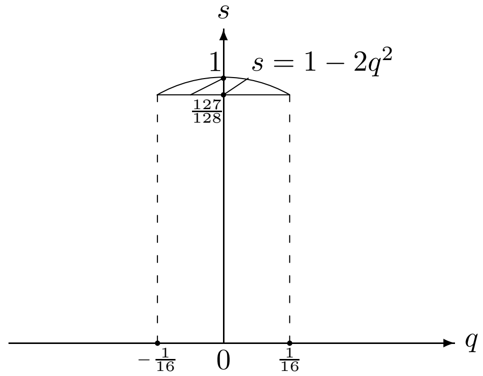

Figure 4. The (q, s) region.

From $y = \gamma_1 + \gamma_2$ and $x = \sqrt{\gamma_1\gamma_2}$ we have

$$\gamma_1 = \tfrac{1}{2}\big(y - \sqrt{y^2 - 4x^2}\big) \qquad \gamma_2 = \tfrac{1}{2}\big(y + \sqrt{y^2 - 4x^2}\big). \tag{26}$$

This implies that γ_2 is a decreasing function of x when y is definite, so the maximum of γ_2 should occur in the curve

$$\tfrac{7}{8} - qx + sx^2 = y \tag{27}$$

that is, the maximum of γ_2 must be the point such that $\frac{\mathrm{d}\gamma_2}{\mathrm{d}x} = \frac{\partial\gamma_2}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial\gamma_2}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} = 0$; that is

$$x_1 = \frac{\frac{7}{2}s + q^2 - 4 + \sqrt{\left(\frac{7}{2}s + q^2 - 4\right)^2 - 14sq^2}}{4sq} \qquad y_1 = \frac{7}{8} - qx_1 + sx_1^2.$$

Thus, the maximum of γ_2 in the plane $\gamma_2 = \gamma_3$ is

$$\gamma_2 = \tfrac{1}{2}\left\{\tfrac{7}{8} - qx_1 + sx_1^2 + \sqrt{\left(\tfrac{7}{8} - qx_1 + sx_1^2\right)^2 - 4x_1^2}\right\} \tag{28}$$

where

$$x_1 = \frac{\frac{7}{2}s + q^2 - 4 + \sqrt{\left(\frac{7}{2}s + q^2 - 4\right)^2 - 14sq^2}}{4sq}. \tag{29}$$

Let

$$w = w(q, s) = \tfrac{7}{8} - qx_1 + sx_1^2 \qquad v = v(q, s) = x_1 \tag{30}$$

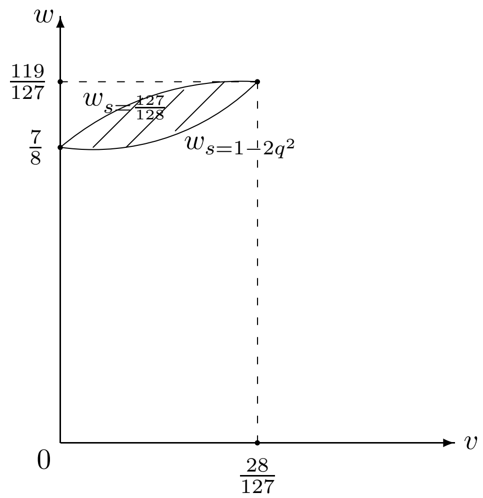

Figure 5. The (v, w) region.

then $w_{s=1-2q^2} = \frac{9}{16}\sqrt{49+32v^2} - \frac{49}{16}$ when $s = 1-2q^2$;

$$v_{s=\frac{127}{128}} = \frac{q^2 - \frac{135}{256} + \sqrt{q^4 - \frac{1913}{128}q^2 + \left(\frac{135}{256}\right)^2}}{\frac{127}{32}q}$$

and

$$w_{s=\frac{127}{128}} = \frac{7}{8} - \frac{q^2 - \frac{135}{256} + \sqrt{q^4 - \frac{1913}{128}q^2 + \left(\frac{135}{256}\right)^2}}{\frac{127}{32}} + \frac{8\left[q^2 - \frac{135}{256} + \sqrt{q^4 - \frac{1913}{128}q^2 + \left(\frac{135}{256}\right)^2}\right]^2}{127q^2}$$

when $s = \frac{127}{128}$. The (v, w) region corresponding to the (q, s) region in figure 4 is depicted in figure 5. Because γ_2 is a decreasing function of v while w is definite, the maximum of γ_2 must be in the left boundary curve $w_{s=\frac{127}{128}}$ in the v–w plane corresponding to the boundary $s = \frac{127}{128}$ in the q–s plane. By $\frac{d\gamma_2}{dq} < 0$, the maximum of γ_2 should be at the point

$$q = -\frac{1}{16} \qquad s = \frac{127}{128}. \tag{31}$$

The exact maximum of γ_2 is

$$\gamma_2 \equiv \gamma(|h_{00110011}\rangle) = \gamma(|h_{11000011}\rangle) = \frac{112}{127} \tag{32}$$

$$\gamma_1 \equiv \gamma(|h_{01000000}\rangle) = \frac{7}{127}. \tag{33}$$

So we do find an exact solution of achievable efficiencies $\gamma_1, \gamma_2, \gamma_3$ satisfying $\gamma_2 = \gamma_3$, and prove that the maximum $\gamma_2 + \gamma_3$ must be greater than or equal to $\frac{224}{127}$.

3. Exact achievable efficiencies for probabilistically cloning the states of [15]

In this section, we will give the exact achievable efficiencies for probabilistically cloning the states in the second example of [15].

In [15], the probabilistic cloning quantum states are

$$|h_1\rangle = |h_{0010}\rangle \equiv \tfrac{1}{2}[|00\rangle + |01\rangle - |10\rangle + |11\rangle] \tag{34}$$

$$|h_2\rangle = |h_{0101}\rangle \equiv \tfrac{1}{2}[|00\rangle - |01\rangle + |10\rangle - |11\rangle] \tag{35}$$

$$|h_3\rangle = |h_{1001}\rangle \equiv \tfrac{1}{2}[-|00\rangle + |01\rangle + |10\rangle - |11\rangle]. \tag{36}$$

We can build probabilistic cloning machines with different cloning efficiencies for each of the states (34)–(36). Let $\gamma_1 \equiv \gamma(|h_{0010}\rangle)$, $\gamma_2 \equiv \gamma(|h_{0101}\rangle)$, $\gamma_3 \equiv \gamma(|h_{1001}\rangle)$ be the achievable efficiencies, and $|P^{(1)}\rangle$, $|P^{(2)}\rangle$, $|P^{(3)}\rangle$ be normalized states of the flag P. P_{ij} denotes the inner product between $|P_i\rangle$ and $|P_j\rangle$, $i, j = 1, 2, 3$. Clearly, $|P_{ij}| \leqslant 1$. Suppose

$$X^{(1)} = \begin{pmatrix} \langle h_1|h_1\rangle & \langle h_1|h_2\rangle & \langle h_1|h_3\rangle \\ \langle h_2|h_1\rangle & \langle h_2|h_2\rangle & \langle h_2|h_3\rangle \\ \langle h_3|h_1\rangle & \langle h_3|h_2\rangle & \langle h_3|h_3\rangle \end{pmatrix}$$

$$X_P^{(2)} = \begin{pmatrix} \langle h_1|h_1\rangle^2 P_{11} & \langle h_1|h_2\rangle^2 P_{12} & \langle h_1|h_3\rangle^2 P_{13} \\ \langle h_2|h_1\rangle^2 P_{21} & \langle h_2|h_2\rangle^2 P_{22} & \langle h_2|h_3\rangle^2 P_{23} \\ \langle h_3|h_1\rangle^2 P_{31} & \langle h_3|h_2\rangle^2 P_{32} & \langle h_3|h_3\rangle^2 P_{33} \end{pmatrix}$$

$$\sqrt{\Gamma} = \begin{pmatrix} \sqrt{\gamma_1} & 0 & 0 \\ 0 & \sqrt{\gamma_2} & 0 \\ 0 & 0 & \sqrt{\gamma_3} \end{pmatrix}$$

then

$$X^{(1)} - \sqrt{\Gamma} X_P^{(2)} \sqrt{\Gamma^+} = \begin{pmatrix} 1-\gamma_1 & -\frac{1}{2} - \frac{\sqrt{\gamma_1\gamma_2}}{4} P_{12} & -\frac{1}{2} - \frac{\sqrt{\gamma_1\gamma_3}}{4} P_{13} \\ -\frac{1}{2} - \frac{\sqrt{\gamma_1\gamma_2}}{4} P_{12}^* & 1-\gamma_2 & 0 \\ -\frac{1}{2} - \frac{\sqrt{\gamma_1\gamma_3}}{4} P_{13}^* & 0 & 1-\gamma_3 \end{pmatrix}.$$

Theorem 2 of [13] provides us with inequalities

$$1 - \gamma_1 \geqslant 0 \tag{37}$$

$$(1-\gamma_1)(1-\gamma_2) - \left|\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\gamma_1\gamma_2} P_{12}\right|^2 \geqslant 0 \tag{38}$$

$$(1-\gamma_1)(1-\gamma_2)(1-\gamma_3) - (1-\gamma_3)\left|\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\gamma_1\gamma_2} P_{12}\right|^2 - (1-\gamma_2)\left|\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\gamma_1\gamma_3} P_{13}\right|^2 \geqslant 0 \tag{39}$$

which allow us to derive achievable efficiencies for the probabilistic cloning process. According to the rule specified in [15] the overall probability (score) of success with the help of probabilistic cloning is given by

$$\begin{aligned} p_2 &= p_{\text{success}} + (1 - p_{\text{success}}) \left[p_{0010} + (1 - p_{0010}) \frac{1}{16} \right] \\ &= \frac{\gamma_1 + \gamma_2 + \gamma_3}{3} + \left(1 - \frac{\gamma_1 + \gamma_2 + \gamma_3}{3}\right) \left[\frac{1-\gamma_1}{3 - \gamma_1 - \gamma_2 - \gamma_3} + \left(1 - \frac{1-\gamma_1}{3-\gamma_1-\gamma_2-\gamma_3}\right) \frac{1}{16} \right] \\ &= [6 + 5(\gamma_2 + \gamma_3)]/16. \end{aligned} \tag{40}$$

From the above equation we know that we should find the maximum of $\gamma_2 + \gamma_3$ satisfying equations (37)–(39).

Our immediate goal is to prove that the maximum of $\gamma_2 + \gamma_3$ must be greater than or equal to $8/7$. For this purpose we discuss the problem in the plane $\gamma_2 = \gamma_3$. In this plane equation (39) becomes

$$(1-\gamma_1)(1-\gamma_2) - \left|\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\gamma_1\gamma_2}P_{12}\right|^2 - \left|\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\gamma_1\gamma_2}P_{13}\right|^2 \geqslant 0. \tag{41}$$

Let

$$\begin{array}{lll} P_{12} = a + b\mathrm{i} & P_{13} = c + d\mathrm{i} & q = \frac{1}{4}(a+c) \\ s = 1 - \frac{1}{16}(a^2+b^2+c^2+d^2) & x = \sqrt{\gamma_1\gamma_2} & y = \gamma_1 + \gamma_2. \end{array} \tag{42}$$

Then equation (41) can be rewritten concisely as

$$\tfrac{1}{2} - qx + sx^2 \geqslant y. \tag{43}$$

Obviously

$$\tfrac{1}{2} - qx + sx^2 \geqslant y \geqslant 2x \geqslant 0. \tag{44}$$

Here $y = \frac{1}{2} - qx + sx^2$ and $y = 2x$ have one intersection point

$$x_0 = \frac{2+q-\sqrt{(2+q)^2-2s}}{2s} \qquad y_0 = 2x_0. \tag{45}$$

The proof is as follows. The intersection points of $y = \frac{1}{2} - qx + sx^2 = \frac{1}{2} - \frac{1}{4}(c+a)x + \left[1 - \frac{1}{16}(a^2+b^2+c^2+d^2)\right]x^2$ and $y = 2x$ are

$$x_0 = \frac{2+q\pm\sqrt{(2+q)^2-2s}}{2s} \qquad y_0 = 2x_0.$$

From $|P_{12}| \leqslant 1$ and $|P_{13}| \leqslant 1$ it is seen $|a+c| \leqslant 2$ and $0 \leqslant a^2+b^2+c^2+d^2 \leqslant 2$, which imply that

$$-\tfrac{1}{2} \leqslant q \leqslant \tfrac{1}{2} \qquad \tfrac{7}{8} \leqslant s \leqslant 1 \tag{46}$$

thus

$$x_0 = \frac{2+q+\sqrt{(2+q)^2-2s}}{2s} > 1$$

which contradicts $x = \sqrt{\gamma_1\gamma_2} \leqslant 1$. Therefore $y = \frac{1}{2} - qx + sx^2$ and $y = 2x$ have one intersection point

$$x_0 = \frac{2+q-\sqrt{(2+q)^2-2s}}{2s} \qquad y = 2x_0.$$

The region in the x–y plane governed by equation (44) is shown in figure 6, where x must satisfy

$$0 \leqslant x \leqslant \frac{2+q-\sqrt{(2+q)^2-2s}}{2s} = x_0. \tag{47}$$

Immediately

$$\frac{\partial x_0}{\partial q} = \frac{1}{2s}\left[1 - \frac{2+q}{\sqrt{(2+q)^2-2s}}\right] \leqslant 0.$$

It follows that when s is definite x_0 is a decreasing function of q. If q is definite (i.e. $a + c = k$ is definite), then the maximum s is to make $a^2+b^2+c^2+d^2 = (a+c)^2+b^2+d^2-2ac$ minimum, which implies $b = d = 0$ and $ac = \frac{(a+c)^2}{4}$. Therefore the curve of maximum s is $s = 1 - \frac{1}{2}q^2$ when q is definite. While s minimum is to make $a^2+b^2+c^2+d^2$ maximum,

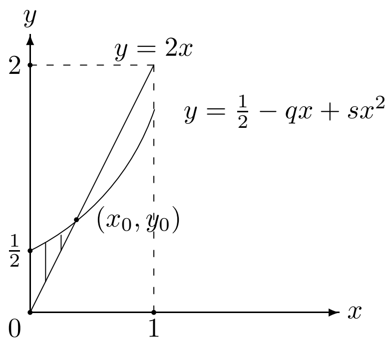

Figure 6. The (x, y) region. Note $\frac{1}{2} - q + s \leqslant 2$.

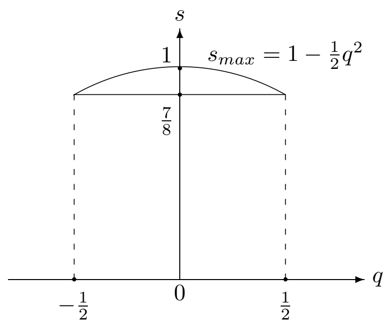

Figure 7. The (q, s) region.

so minimum s is $s = \frac{7}{8}$ in the case q is definite. The boundary of s and q is illustrated in figure 7.

By $x = \sqrt{\gamma_1\gamma_2}$ and $y = \gamma_1 + \gamma_2$ we get

$$\gamma_1 = \tfrac{1}{2}\big(y - \sqrt{y^2 - 4x^2}\big) \qquad \gamma_2 = \tfrac{1}{2}\big(y + \sqrt{y^2 - 4x^2}\big). \tag{48}$$

It follows that if y is definite, the smaller x is, the bigger γ_2 is, so the maximum of γ_2 should take place in the curve

$$\tfrac{1}{2} - qx + sx^2 = y \tag{49}$$

that is, the maximum of γ_2 must be the point such that $\frac{\mathrm{d}\gamma_2}{\mathrm{d}x} = \frac{\partial\gamma_2}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial\gamma_2}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} = 0$, that is

$$x_1 = \frac{2s + q^2 - 4 + \sqrt{(2s + q^2 - 4)^2 - 8sq^2}}{4sq} \qquad y_1 = \frac{1}{2} - qx_1 + sx_1^2.$$

Thus, the maximum of γ_2 in the plane $\gamma_2 = \gamma_3$ is

$$\gamma_2 = \tfrac{1}{2}\Big\{\tfrac{1}{2} - qx_1 + sx_1^2 + \sqrt{\big(\tfrac{1}{2} - qx_1 + sx_1^2\big)^2 - 4x_1^2}\Big\} \tag{50}$$

where

$$x_1 = \frac{2s + q^2 - 4 + \sqrt{(2s + q^2 - 4)^2 - 8sq^2}}{4sq}. \tag{51}$$

Next we derive the maximum of γ_2. Let

$$w = w(q, s) = \tfrac{1}{2} - qx_1 + sx_1^2 \qquad v = v(q, s) = x_1. \tag{52}$$

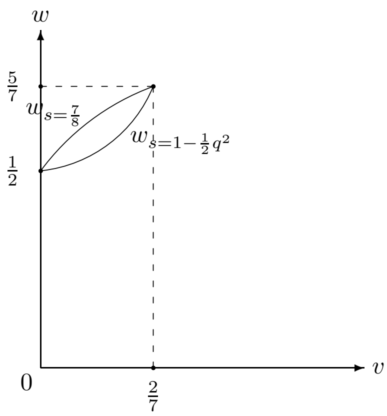

Figure 8. The (v, w) region.

Now we change the (q, s) region to the (v, w) region. When $s = 1 - \frac{1}{2}q^2$, then $v = -\frac{q}{2-q^2}$ and $w = \frac{1}{2} + \frac{3q^2}{2(2-q^2)}$. From $v = -\frac{q}{2-q^2}$, $|q| \leqslant \frac{1}{2}$ and $v = x_1 \geqslant 0$ we know that $q = \frac{1-\sqrt{1+8v^2}}{2v}$, $0 \leqslant v \leqslant \frac{2}{7}$. Hence $w_{s=1-\frac{q^2}{2}} = -\frac{1}{4} + \frac{3}{4}\sqrt{1+8v^2}$ and $0 \leqslant v_{s=1-\frac{q^2}{2}} \leqslant \frac{2}{7}$ in the case $s = 1 - \frac{1}{2}q^2$. Note that

$$v_{s=\frac{7}{8}} = \frac{-\frac{9}{4} + q^2 + \sqrt{\left(q^2 - \frac{9}{4}\right)^2 - 7q^2}}{\frac{7}{2}q}$$

and

$$w_{s=\frac{7}{8}} = \frac{1}{2} - \frac{-\frac{9}{4} + q^2 + \sqrt{\left(q^2 - \frac{9}{4}\right)^2 - 7q^2}}{\frac{7}{2}} + \frac{7}{8}\left(\frac{-\frac{9}{4} + q^2 + \sqrt{\left(q^2 - \frac{9}{4}\right)^2 - 7q^2}}{\frac{7}{2}q}\right)^2$$

if $s = \frac{7}{8}$. The (v, w) region corresponding to the (q, s) region is shown in figure 8.

Since γ_2 is a decreasing function of v as w is definite, from equation (50) we obtain that the maximum of γ_2 must appear in the left boundary curve $w_{s=\frac{7}{8}}$ in the v–w plane corresponding to the boundary $s = \frac{7}{8}$ in the q–s plane. It can be seen that

$$\frac{\mathrm{d}\gamma_2}{\mathrm{d}q} < 0 \tag{53}$$

while $s = \frac{7}{8}$. Therefore the maximum of γ_2 should exist at the point

$$q = -\frac{1}{2} \qquad s = \frac{7}{8}. \tag{54}$$

The exact maximum of γ_2 is

$$\gamma_2 = \frac{4}{7} \simeq 0.571\,43 \tag{55}$$

and

$$\gamma_1 = \frac{1}{7} \simeq 0.142\,86. \tag{56}$$

It is clear that our analytic solution is better as compared with the numerical result

$$\gamma_1 = 0.141\,65 \qquad \gamma_2 = \gamma_3 = 0.571\,22 \tag{57}$$

of [15], since equations (55) and (56) are exact solutions. Evidently the maximum of $\gamma_2 + \gamma_3$ should be greater than or equal to $\frac{8}{7}$ although we guess that $\frac{8}{7}$ should be the maximum of $\gamma_2 + \gamma_3$.

However if we make $\gamma_1 + \gamma_2$ to be maximum, under the condition $\gamma_2 = \gamma_3$, it is not difficult to obtain that the probability of cloning success is, on average,

$$P_{\text{success}} = \gamma_1 = \gamma_2 = \gamma_3 = 1 - \frac{2\sqrt{2}+1}{7} \simeq 0.453\,08. \tag{58}$$

We have constructed the quantum logic network for probabilistically cloning the states [15] in [16].

In summary, we give achievable efficiencies for probabilistically cloning the quantum states used in implemented tasks for which cloning provides some enhancement in performance, and present an example of quantum computational tasks whose performance is enhanced if we distribute quantum information using quantum cloning. We hope our result will be helpful in the quantum information processing.

Acknowledgments

This work was supported by National Natural Science Foundation of China under grant no 10271081 and Hebei Natural Science Foundation under grant no 101094.

References

[1] Wootters W K and Zurek W H 1982 *Nature* **299** 802
[2] Dieks D 1982 *Phys. Lett.* A **92** 271
[3] Bužek V and Hillery M 1996 *Phys. Rev.* A **54** 1844
[4] Gisin N and Massar S 1997 *Phys. Rev. Lett.* **79** 2153
[5] Gisin N 1998 *Phys. Lett.* A **242** 1
[6] Werner R F 1998 *Phys. Rev.* A **58** 1827
[7] Keyl M and Werner R F 1999 *J. Math. Phys.* **40** 3283
[8] Bruß D and Macchiavello C 1999 *Phys. Lett.* A **253** 249
[9] Bruß D, Ekert A and Macchiavello C 1998 *Phys. Rev. Lett.* **81** 2598
[10] Bužek V, Hillery M and Bednik R 1998 *Acta Phys. Slovaca* **48** 177
[11] Nicolas J C 2000 *J. Mod. Opt.* **47** 187
[12] Bužek V, Braunstein S, Hillery M and Bruß D 1997 *Phys. Rev.* A **56** 3446
[13] Duan L M and Guo G C 1998 *Phys. Rev. Lett.* **80** 4999
[14] Duan L M and Guo G C 1999 *Commun. Theor. Phys.* **31** 223
[15] Galvão E F and Hardy L 2000 *Phys. Rev.* A **62** 022301
[16] Gao T, Yan F and Wang Z 2003 *Preprint* quant-ph/0308036